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General solution of the particle momentum equation in unsteady Stokes flows

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The general solution of the particle momentum equation for unsteady Stokes flows is obtained analytically. The method used to obtain the solution consists of applying a fractional-differential operator to the first-order integro-differential equation of motion in order to transform the original equation into a second-order non-homogeneous equation, and then solving this last equation by the method of variation of parameters. The fractional differential operator consists of a three-time-scale linear operator that stretches the order of the Riemann–Liouville fractional derivative associated with the history term in the equation of motion. In order to illustrate the application of the general solution to particular background flow fields, the particle velocity is calculated for three specific flow configurations. These flow configurations correspond to the gravitationally induced motion of a particle through an otherwise quiescent fluid, the motion of a particle caused by a background velocity field that accelerates linearly in time, and the motion of a particle in a fluid that undergoes an impulsive acceleration. The analytical solutions for these three specific cases are analysed and compared to other solutions found in the literature.

1. Introduction

The description of the motion of small particles in a viscous flow is of fundamental importance in many engineering and scientific fields. The vast majority of the studies in the field of multiphase flows are conducted numerically, where the particle equation of motion is either solved through iterative schemes, or is simplified to allow for fast computations. In either situation, the outcome of the numerical studies is compromised, since computational costs need to be balanced against unrealistic solutions. The most commonly used strategy is to neglect troublesome terms in the equation of motion and solve exactly or numerically the remaining simplified equation. This option proves to be unsatisfactory for many important flows, especially when the unsteadiness of the flow field near the particle plays an important role in the balance of forces acting on the particle. In this work, we solve the particle equation of motion exactly, overcoming the most fundamental problems of computing particle velocities in unsteady viscous flows.

Among the earliest attempts to describe the motion of a rigid particle in a viscous flow is the equation derived by Oseen (1927), based upon the works of Boussinesq (1885) and Basset (1888). Because of these fundamental contributions, the equation of motion in uniform flows is sometimes called the BBO equation (Basset–Boussinesq–Ossen). Vojir & Michaelides (1994) pointed out that the basic form of the equation was first derived by Boussinesq in 1885. The very common terminology of *Basset equation* or *Basset force* would then be inappropriate. In this work, we refer to the equation as the *particle momentum equation* and to the *Boussinesq–Basset* term in the equation as the *history term*.

Several forms of the equation of motion for a small particle moving in unsteady flows have been studied in this century (Tchen 1947; Corrsin & Lumley 1956; Maxey & Riley 1983), but few attempts have been made to solve the equation of motion for the velocity of the particle. Also, and most surprisingly, no previous attempts to describe generally the velocity of the particle as a function of the background flow characteristics have been conducted. The basic form of the equation of motion in uniform flows has been known for more than a century, but the BBO equation with unsteady forcing derived by Tchen (1947) was never analytically solved in a general form. The objective of this work is to develop a procedure to solve the equation exactly (to the approximations made in the derivation of the equation) for a generic uniform flow field characterized by a background velocity $\boldsymbol{U}(t) \neq 0$. Following this objective, the equation of motion for the limit of infinitesimal particle Reynolds number is solved analytically. The focus of the present work is the mathematical method used to find the analytical solution of the equation of motion, and not the correction of this equation for the case of a finite particle Reynolds number. However, the analytical solution of this classical problem in fluid dynamics should be valuable as a limiting case for higher-order expansions in the particle Reynolds number.

In §2, a fractional differential operator is applied to a dimensionless form of the particle equation of motion. This procedure yields a second-order ordinary differential equation on the relative particle-to-fluid velocity. During the derivation of the explicit equation, attention is given to the identification of the terms in this equation with the original forces that generate them. The resulting non-homogenous second-order equation is then solved exactly in §3 using variation of parameters. In §4 we treat three specific cases using the general solutions derived in §3. The gravitationally induced motion of a sphere in a quiescent fluid is studied in §4.1. This is the only case known to the authors for which the particle momentum equation has been solved analytically in previous works (see Boggio 1927; Sy, Taunton & Lightfoot 1970; Clift, Grace & Weber 1978). The procedure used by previous authors consists of working in the Laplace space and inverting back to the time space only the simplified equation corresponding to the background flow velocity $\boldsymbol{U}(t) = 0$. An alternative treatment was suggested by Konopliv (1971), who Laplace-transformed a simplified version of the momentum equation into an explicit second-order differential equation. Konopliv's procedure was extended by Michaelides (1992) to a general flow field $\boldsymbol{U}(t)$.

In §4.2, we analyse the motion of a particle in a viscous fluid that accelerates linearly in time. This problem is shown to be equivalent to the gravitationally induced motion in a quiescent flow if a non-inertial reference frame moving with the flow velocity is chosen by the observer. The equivalence of the two motions should be obvious from the solution for the particle velocity in an inertial reference frame, thus providing a good test for the general solution of motion given in §3.

Hinch (1975) studied the response of a particle that undergoes an impulsive start in connection with the Langevin equation. The impulsive motion of a particle in a viscous fluid has been treated recently by Felderhof (1991) through an asymptotic analysis. A similar problem is treated in §4.3, where the problem under consideration is that of a spherical particle that is initially stationary in a fluid that undergoes an impulsive start to a constant velocity. The time derivative of the background flow velocity in this case is a delta function acting at $t = 0^{+}$.

2. Equation of motion for small particles in unsteady flows

The viscous flow motion of small particles is described by the equation derived by Maxey & Riley (1983). The resulting equation for the motion of a small rigid particle subjected to non-uniform flows presents several additional terms in relation to the BBO equation and, due to its completeness, it is the starting point of our derivation. Maxey & Riley presented the particle equation of motion as (Maxey & Riley 1983; Maxey 1987)

$$
\begin{aligned}
m_p \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = {} & m_f \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} - m_f \frac{\mathrm{d}(\boldsymbol{V}-\boldsymbol{U})}{2\mathrm{d}t} - 6\pi a\mu(\boldsymbol{V}-\boldsymbol{U}) \\
& + (m_p - m_f)\,\boldsymbol{g} - \frac{6\pi a^2\mu}{(\pi\nu)^{1/2}}\left(\int_0^t \frac{\mathrm{d}(\boldsymbol{V}-\boldsymbol{U})}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(t-\sigma)^{1/2}} + \frac{\boldsymbol{V}(0)-\boldsymbol{U}(0)}{t^{1/2}}\right) \\
& + m_f \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^2\nabla^2\boldsymbol{U}}{20}\right) + \pi a^3\mu\nabla^2\boldsymbol{U} + \frac{a^4\mu\pi^{1/2}}{\nu^{1/2}}\left(\int_0^t \frac{\mathrm{d}(\nabla^2\boldsymbol{U})}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(t-\sigma)^{1/2}} + \frac{\nabla^2\boldsymbol{U}(0)}{t^{1/2}}\right),
\end{aligned}
\qquad (2.1)
$$

where $\boldsymbol{V}$ and $\boldsymbol{U}$ are the particle and fluid velocities, respectively, t is time, a is the radius of the particle, m_p is the mass of the particle, m_f is the displaced fluid mass, $\boldsymbol{g}$ is the gravitational acceleration vector, and μ are ν the dynamic and kinematic viscosities, respectively. The fluid velocity $\boldsymbol{U}$ in equation (2.1) is the fluid velocity at the current position of the particle. The terms in brackets containing the integral from zero to t on the right-hand side of equation (2.1) are the history terms responsible for the modification of the drag due to the unsteadiness of the near flow field as the particle moves through the fluid. The last term in each bracket accounts for non-zero initial relative velocity (Maxey 1987). In order to simplify the following derivation, the particle and the fluid are assumed to have initially the same velocity (zero initial Reynolds number). However, the analysis can be easily extended to non-zero initial relative velocity.

The nonlinearity of equation (2.1) is related to the fact that the substantial derivative $\mathrm{D}/\mathrm{D}t$ and the terms involving the fluid velocity must be evaluated at the position of the particle. It is important to emphasize the distinction between the substantial derivative following a fluid particle ($\mathrm{D}/\mathrm{D}t$) and the substantial derivative following the rigid particle ($\mathrm{d}/\mathrm{d}t$), not only because the values of the derivatives are not equal, but also because equation (2.1) is not limited to small *flow Reynolds number* $Re_f = U_o L/\nu$, where U_o is the characteristic flow velocity, and L is characteristic length of the background flow. This feature of the equation allows calculation of, for example, the motion of a small particle subjected to a turbulent background flow if the Kolmogorov length scale is much larger than the radius of the particle. Nonetheless, equation (2.1) is only valid for small *particle Reynolds number* ($Re_p = |\boldsymbol{U}-\boldsymbol{V}|\,a/\nu \quad 1$), small *shear Reynolds number* ($Re_s = U_o a^2/\nu L \quad 1$), and applicable only for a small particle so that $a/L \quad 1$. If the background flow field under consideration is uniform, both substantial derivatives are identically equal, and the restriction of small shear Reynolds number is relaxed. For the cases where the background flow field has negligible convective acceleration (Stokes flow) approximating the substantial derivative $\mathrm{D}/\mathrm{D}t$ as $\mathrm{d}/\mathrm{d}t$ is exact to the order of approximation made in the derivation of (2.1). In this work, we concentrate on the exact analytical solution of the particle momentum equation for uniform background flows, but the form of the general solution derived in this work is valid whenever $\mathrm{D}/\mathrm{D}t$ can be approximated by $\mathrm{d}/\mathrm{d}t$.

The terms involving the Laplacian of the flow velocity in the third line of (2.1) represent the Faxén corrections for the non-uniformity of the flow field. These terms are only relevant for background flow fields that present strong velocity gradients (significant velocity gradients on the scale of the particle radius). For uniform flows, the Faxén corrections are zero. Equation (2.1) is the original form of the equation derived by Maxey & Riley (1983) and Maxey (1987). Auton, Hunt & Prud'homme (1988) studied the problem of inviscid flow over a sphere and showed that the second term on the right-hand side of (2.1), the so-called added mass term, should be written as $m_f(\mathrm{d}\boldsymbol{V}/\mathrm{d}t)/2-m_f(\mathrm{D}\boldsymbol{U}/\mathrm{d}t)/2$. Since the present work is concerned with unsteady Stokes flows, the suggested modification is not relevant.

For zero initial particle Reynolds number, an equation of motion is obtained from the non-dimensionalization of equation (2.1):

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\hat{t}}=\alpha\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}-\alpha\frac{\mathrm{d}(\boldsymbol{v}-\boldsymbol{u})}{2\mathrm{d}\hat{t}}-(\boldsymbol{v}-\boldsymbol{u})+(1-\alpha)\frac{\tau_p\boldsymbol{g}}{U_o}-\left(\frac{9\alpha}{2\pi}\right)^{1/2}\int_0^{\hat{t}}\frac{\mathrm{d}(\boldsymbol{v}-\boldsymbol{u})}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}} \quad (2.2)$$

In equation (2.2), α is the fluid-to-particle density ratio, τ_p is a particle characteristic time given by $2\rho_p a^2/9\mu$, $\hat{t}$ is the dimensionless time, $\boldsymbol{v}$ and $\boldsymbol{u}$ are the dimensionless particle and fluid velocities, respectively. Time, velocity, and length are non-dimensionalized by τ_p, U_o, and L, respectively.

Defining $\boldsymbol{w}$ as the relative velocity $(\boldsymbol{v}-\boldsymbol{u})$, equation (2.2) can be rewritten as

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\hat{t}}=(\alpha-1)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}}-\alpha\frac{\mathrm{d}\boldsymbol{w}}{2\mathrm{d}\hat{t}}-\boldsymbol{w}-\left(\frac{9\alpha}{2\pi}\right)^{1/2}\int_0^{\hat{t}}\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}}+(1-\alpha)\frac{\tau_p\boldsymbol{g}}{U_o}. \quad (2.3)$$

In order to later recognize the contribution of the different terms in equation (2.3), each term is multiplied by a binary coefficient (P^* = *pressure*, H^* = *history*, M^* = *virtual mass*, D^* = *drag*, and G^* = *gravity*), with possible values of 0 or 1, depending on the consideration or not of the corresponding forces. Equation (2.3) then becomes

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\hat{t}}=(\alpha P^*-1)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}}-M^*\alpha\frac{\mathrm{d}\boldsymbol{w}}{2\mathrm{d}\hat{t}}-D^*\boldsymbol{w}-H^*\left(\frac{9\alpha}{2\pi}\right)^{1/2}$$
$$\times\int_0^{\hat{t}}\frac{\mathrm{d}\boldsymbol{w}(\sigma)}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}}+G^*(1-\alpha)\frac{\tau\boldsymbol{g}}{U_o}. \quad (2.4)$$

The following coefficients can also be defined:

$$D=D^*, \quad (2.5)$$

$$P=1-\alpha P^*, \quad (2.6)$$

$$M=M^*\alpha/2, \quad (2.7)$$

$$H=H^*(9\alpha/2\pi)^{1/2}, \quad (2.8)$$

$$\boldsymbol{G}=G^*(1-\alpha)\tau_p\boldsymbol{g}/U_o, \quad (2.9)$$

$$k=1/(1+M). \quad (2.10)$$

In terms of the new coefficients, the equation of motion for a particle is written as

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\hat{t}}+Dk\boldsymbol{w}+Hk\left\{\int_0^{\hat{t}}\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{(\hat{\tau}-\sigma)^{1/2}}\right\}=\boldsymbol{G}k-Pk\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}}. \quad (2.11)$$

We identify the history term (in curly brackets) as $\pi^{1/2}$ times the Riemann–Liouville half-derivative of the relative velocity $\boldsymbol{w}$. The Riemann–Liouville fractional derivative of order n of $f(\hat{t})$ is defined as (Zayed 1996)

$$\frac{\mathrm{d}^n f(\hat{t})}{\mathrm{d}\hat{t}^n} = \begin{cases} \dfrac{1}{\Gamma(-n)} \displaystyle\int_{-\infty}^{\hat{t}} (\hat{t}-\sigma)^{-n-1} f(\sigma)\,\mathrm{d}\sigma, & n < 0 \\ \dfrac{1}{\Gamma(m-n)} \dfrac{\mathrm{d}^m}{\mathrm{d}\hat{t}^m}\left(\displaystyle\int_{-\infty}^{\hat{t}} (\hat{t}-\sigma)^{m-n-1} f(\sigma)\,\mathrm{d}\sigma \right), & n \geqslant 0, \end{cases} \tag{2.12}$$

where $\Gamma(z)$ is the gamma (generalized factorial) function of z, and $m-1 \leqslant n < m$, $m = 1, 2, 3, \ldots$.

In definition (2.12), we take $n = -1/2$ and $f(\hat{t}) = \mathrm{d}\boldsymbol{w}/\mathrm{d}\hat{t}$, and using the property $\mathrm{d}^{-1/2}/\mathrm{d}\hat{t}^{-1/2}(\mathrm{d}\boldsymbol{w}/\mathrm{d}\hat{t}) = \mathrm{d}^{1/2}\boldsymbol{w}/\mathrm{d}\hat{t}^{1/2}$, we reduce equation (2.11) to

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\hat{t}} + Hk\pi^{1/2}\frac{\mathrm{d}^{1/2}\boldsymbol{w}}{\mathrm{d}\hat{t}^{1/2}} + Dk\boldsymbol{w} = \boldsymbol{G}k - Pk\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}}. \tag{2.13}$$

Equation (2.13) makes explicit the characteristic feature of the equation of motion that makes it hard to solve. The equation is characterized by three terms in $\boldsymbol{w}$ which are associated with three different time scales, including a non-integer scale. In order to circumvent this difficulty, we apply the following linear operator to (2.13):

$$\Psi = \frac{\mathrm{d}}{\mathrm{d}\hat{t}} - Hk\pi^{1/2}\frac{\mathrm{d}^{1/2}}{\mathrm{d}\hat{t}^{1/2}} + Dk, \tag{2.14}$$

resulting in

$$\frac{\mathrm{d}^2\boldsymbol{w}}{\mathrm{d}\hat{t}^2} + (2Dk - H^2\pi k^2)\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\hat{t}} + D^2k^2\boldsymbol{w} = \boldsymbol{G}k^2D - \frac{Hk^2}{\hat{t}^{1/2}}\left(\boldsymbol{G} - P\left.\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}}\right|_{\hat{t}=0}\right)$$
$$- Pk\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}\hat{t}^2} - PDk^2\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\hat{t}} + HPk^2\left(\int_0^{\hat{t}} \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}\sigma^2}\frac{\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}}\right). \tag{2.15}$$

The second term on the right-hand-side of (2.15) results from the definition (2.12) – the contribution to the half-derivative of $f(\hat{t})$ of the time integral from $-\infty$ to 0 is equal to $f(0)/(\pi\hat{t})^{1/2}$.

When the binary coefficients (with superscript *) are set to unity, (2.15) is equivalent to the one derived by Michaelides (1992), who Laplace-transformed the equation of motion (2.3) to derive an equivalent second-order differential equation in $\boldsymbol{w}$. The idea of Laplace-transforming the equation of motion to transform the original integro-differential (or non-integer differential) equation into an ordinary differential equation was first suggested by Konopliv (1971). The advantage of equation (2.15) over Michaelides' equation is that it allows the association of individual terms with the forces that generated them. This is important if one wishes to understand the contribution of these terms in the following derivation. It should be noticed that Michaelides' equation already presents an improvement over equation (2.2), since his equation is explicit in $\boldsymbol{w}$, although of higher order. Previously, Boggio (1927), and Sy *et al.* (1970), obtained analytical solutions for the case of $\boldsymbol{u}(\hat{t}) = 0$. Konopliv (1971) Laplace transformed the equation of motion for the case of $\boldsymbol{u}(\hat{t}) = 0$ to obtain a much simpler form of (2.15). Michaelides (1992) extended Konopliv's method for all $\boldsymbol{u}(\hat{t})$, but resorted to numerical methods to solve the resulting second-order explicit differential equation (Konopliv suggested that the equation derived in his short note be solved numerically, but did not present any result). We will show that it is possible to solve analytically (2.15) for a general flow velocity $\boldsymbol{u}(\hat{t})$.

3. Analytical solution of the equation of motion

In order to simplify the notation in the following derivation, we define the following parameters that are functions of α only:

$$b = (2D - H^2\pi k)\,k, \tag{3.1}$$

$$c = D^2k^2, \tag{3.2}$$

$$\Delta = b^2 - 4c, \tag{3.3}$$

$$Q = \tfrac{1}{2}(-b - \Delta^{1/2}), \tag{3.4}$$

$$R = \tfrac{1}{2}(-b + \Delta^{1/2}). \tag{3.5}$$

Equation (2.15) is then rewritten in terms of the parameters defined by (3.1) to (3.5), with dots denoting integer derivatives with respect to $\hat{t}$ or to the dummy variable of integration:

$$\ddot{\boldsymbol{w}} + b\dot{\boldsymbol{w}} + c\boldsymbol{w} = \boldsymbol{G}Dk^2 - \frac{Hk^2[\boldsymbol{G} - P\dot{\boldsymbol{u}}(0)]}{\hat{t}^{1/2}} - Pk\ddot{\boldsymbol{u}} - PDk^2\dot{\boldsymbol{u}} + PHk^2 \int_0^{\hat{t}} \frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}}. \tag{3.6}$$

Equation (3.6) is a second-order ordinary non-homogeneous linear differential equation with constant coefficients. Because the coefficients b and c in the homogeneous part of equation (3.6) are functions of the coefficients D, H and M (this last one through k), the solution for the particle velocity has a nonlinear dependence on these coefficients, making it more difficult to directly associate terms in the solution for the particle velocity with the specific forces that generated them. This result is expected since these forces are dependent on the velocity of the particle and its derivatives in (2.2). The contribution of virtual mass is embedded in the k factor, which varies from zero (infinitely light particle) to one (infinitely heavy particle). The virtual mass effect can be easily assessed by considering solutions with the actual value of k and solutions with k equal to 1 (no virtual mass effect). The influence of the history term can be properly quantified if the general solution for the complete equation of motion is found. This is because the solution of (2.11) neglecting the history term is very easily found through the method of variation of parameters or through the use of an integrating factor. In other words, a complete understanding of the forces in the motion of a small particle through a viscous fluid can be achieved if the solution to the complete equation of motion is found.

Figure 1 shows the coefficients b, c, Δ, k and H for values of the fluid-to-particle density ratio α ranging from 10^{-3} to 10^3. These coefficients play a fundamental role in the mathematical behaviour of the general solution of (3.6), and also help clarify the importance of particular terms in the equation. The coefficients b and c are the coefficients in the characteristic equation associated with (3.6). The 'damping' coefficient b assumes negative values for values of α larger than 4/9. This indicates that the rate of change of the relative acceleration is enhanced by the relative acceleration for values of α in the range $[4/9:\infty)$, and not damped by the relative acceleration as it happens for smaller values of α. The 'spring-constant' coefficient c is equal to k^2, and reflects the importance of the virtual mass for the motion of the particle. As discussed before, the virtual mass coefficient k assumes values in the range $[0:1]$ as the particle

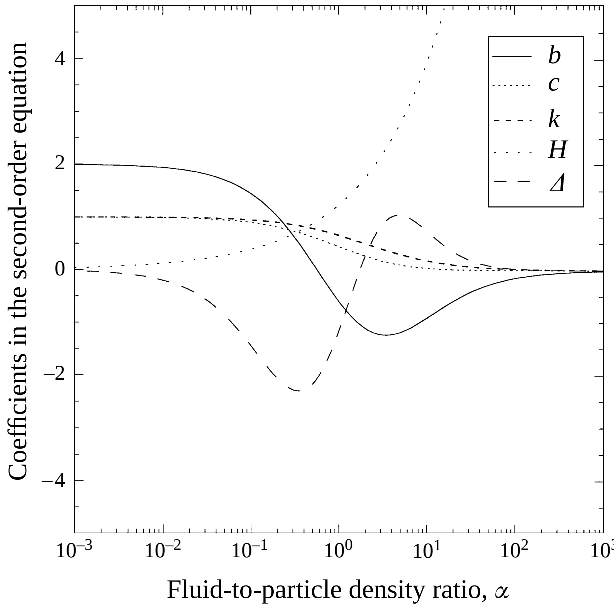

FIGURE 1. Coefficients in the second-order equation of motion.

goes from infinitely heavy to infinitely light. The form of the general solution for equation (3.6) depends on the value of the coefficient Δ, defined by (3.3). The coefficient Δ is negative for a heavy particle and for a particle with density similar to the fluid, but becomes positive for a value of α equal to or greater than 8/5. Clearly, when the density of the fluid is equal to the density of the particle equation (3.6) yields a trivial solution ($\boldsymbol{w} = 0$). The history coefficient H is negligible for a heavy particle but assumes a value that is orders of magnitude larger than all other coefficients in (3.6) for a particle much lighter than the fluid. This observation is of particular interest for the situations where the history drag does not asymptote to zero at initial times, as it is the case for the impulsive start discussed in §4.3. In this situation, the history drag is orders of magnitude larger than the Stokes drag for light particles and short times.

Further analysis of equation (3.6) shows that contributions from the time derivatives of the fluid velocity and from the gravity field are only present in the non-homogeneous part of that equation. The fact that both (2.11) and the operator Ψ are linear makes it possible to maintain the contributions from the background flow in the non-homogeneous part of the equation, allowing a general solution for a generic background flow to be found. The following subsections treat the cases when $\Delta > 0$ ($\alpha > 8/5$), $\Delta = 0$ ($\alpha = 8/5$), and $\Delta < 0$ ($\alpha < 8/5$), respectively.

3.1. *The case* $\Delta > 0$ ($\alpha > 8/5$)

For the case when the fluid-to-particle density ratio is larger than the critical value of 8/5, the characteristic equation associated with the homogeneous part of (3.6) has two distinct and real roots, and the general solution of the homogeneous equation has the form

$$\boldsymbol{w}_h(\hat{t}) = \boldsymbol{A}\,\mathrm{e}^{R\hat{t}} + \boldsymbol{B}\,\mathrm{e}^{Q\hat{t}}, \tag{3.7}$$

with the Wronskian of the two linearly independent solutions given by

$$W_{1,2}(\hat{t}) = -\Delta^{1/2}\,\mathrm{e}^{-b\hat{t}}. \tag{3.8}$$

The general solution for the relative velocity $\boldsymbol{w}(\hat{t})$ when $\Delta > 0$ is found to be

$$\begin{aligned}\boldsymbol{w}(\hat{t}) = {} & \boldsymbol{A}\,\mathrm{e}^{R\hat{t}} + \boldsymbol{B}\,\mathrm{e}^{Q\hat{t}} + \left[1 - \left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erf}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erf}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\right]\boldsymbol{G} \\ & + (1-\alpha)\left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erf}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erf}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\dot{\boldsymbol{u}}(0) \\ & + \frac{(1-\alpha)\,k\,\mathrm{e}^{R\hat{t}}}{\Delta^{1/2}} \int_0^{\hat{t}} \left[\mathrm{e}^{-R\tau}\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k \int_0^{\tau} \frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}} - \ddot{\boldsymbol{u}} - k\dot{\boldsymbol{u}}\right)\right]\mathrm{d}\tau \\ & - \frac{(1-\alpha)\,k\,\mathrm{e}^{Q\hat{t}}}{\Delta^{1/2}} \int_0^{\hat{t}} \left[\mathrm{e}^{-Q\tau}\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k \int_0^{\tau} \frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\hat{t}-\sigma)^{1/2}} - \ddot{\boldsymbol{u}} - k\dot{\boldsymbol{u}}\right)\right]\mathrm{d}\tau. \qquad (3.9)\end{aligned}$$

The two initial conditions associated with equation (3.6) are used to find the constant vectors $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{w}(0) = 0, \qquad (3.10)$$

$$\dot{\boldsymbol{w}}(0) = \boldsymbol{G}k - (1-\alpha)\,k\dot{\boldsymbol{u}}(0). \qquad (3.11)$$

The first initial condition is due to the original assumption of initial particle Reynolds number equal to zero. The second condition is derived directly from (2.11). The constant vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are then obtained as

$$\boldsymbol{A} = \frac{k}{\Delta^{1/2}}\left(1 + \frac{kD}{R}\right)\boldsymbol{G} - \frac{(1-\alpha)\,k}{\Delta^{1/2}}\dot{\boldsymbol{u}}(0), \qquad (3.12)$$

$$\boldsymbol{B} = \frac{-k}{\Delta^{1/2}}\left(1 + \frac{kD}{Q}\right)\boldsymbol{G} + \frac{(1-\alpha)\,k}{\Delta^{1/2}}\dot{\boldsymbol{u}}(0). \qquad (3.13)$$

Thus, the relative velocity of a particle lighter than critical is given by

$$\begin{aligned}\boldsymbol{w}(\hat{t}) = {} & \left[1 + \left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erfc}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erfc}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\right]\boldsymbol{G} \\ & + (1-\alpha)\frac{(\mathrm{e}^{Q\hat{t}} - \mathrm{e}^{R\hat{t}})}{\Delta^{1/2}} k\dot{\boldsymbol{u}}(0) \\ & + (1-\alpha)\left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erf}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erf}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\dot{\boldsymbol{u}}(0) \\ & + \frac{(1-\alpha)\,k\,\mathrm{e}^{R\hat{t}}}{\Delta^{1/2}} \int_0^{\hat{t}} \left[\mathrm{e}^{-R\tau}\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k \int_0^{\tau} \frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}} - \ddot{\boldsymbol{u}} - k\dot{\boldsymbol{u}}\right)\right]\mathrm{d}\tau \\ & - \frac{(1-\alpha)\,k\,\mathrm{e}^{Q\hat{t}}}{\Delta^{1/2}} \int_0^{\hat{t}} \left[\mathrm{e}^{-Q\tau}\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k \int_0^{\tau} \frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}} - \ddot{\boldsymbol{u}} - k\dot{\boldsymbol{u}}\right)\right]\mathrm{d}\tau. \qquad (3.14)\end{aligned}$$

Equation (3.14) is the general solution for the particle velocity in unsteady Stokes or uniform flows for the case when the fluid is more than 8/5 times denser than the particle, and for initial Reynolds number equal to zero. The restriction on the initial Reynolds number is not practically binding because the same solution procedure advanced here can be used to find the solution for different initial Reynolds numbers. Equation (3.14) is of particular interest in marine sciences because it allows the motion of small bubbles that behave as light solid spheres to be described. The presence of surfactants in small sea bubbles may reduce the strength of the internal circulation and allow one to impose the no-slip condition as if the bubbles were rigid spheres (Clift *et al.* 1978). The unsteady motion of these small bubbles is greatly influenced by the history term since the inertia of the bubbles is much smaller than the inertia of the surrounding fluid.

Equation (3.14) relates explicitly the particle velocity to the background flow. We proceed in the next two subsections with the derivation for the other possible values for Δ. Section 4 is concerned with applications of the general solutions to three different unperturbed flow fields $\boldsymbol{u}(\hat{t})$.

3.2. *The case* $\Delta = 0$ ($\alpha = 8/5$)

When the fluid-to-particle density ratio is equal to 8/5 (critical density ratio), the solution to the characteristic equation associated with (3.6) has two identical roots, leading to the following form of the solution for the homogeneous equation:

$$\boldsymbol{w}_h(\hat{t}) = \boldsymbol{A}\,\mathrm{e}^{k\hat{t}} + \boldsymbol{B}\hat{t}\,\mathrm{e}^{k\hat{t}}, \tag{3.15}$$

with an associate Wronskian given by

$$W_{1,2}(\hat{t}) = \mathrm{e}^{2k\hat{t}}. \tag{3.16}$$

The general solution for this critical case is found through the method of variation of parameters after application of the initial conditions (3.10) and (3.11) to yield

$$\begin{aligned}
\boldsymbol{w}(\hat{t}) = {} & \boldsymbol{G}(1-\mathrm{e}^{k\hat{t}}) + [\boldsymbol{G}(1+k) - (1-\alpha)\,k\dot{\boldsymbol{u}}(0)]\,\hat{t}\,\mathrm{e}^{k\hat{t}} \\
& + [(1-\alpha)\,\dot{\boldsymbol{u}}(0) - \boldsymbol{G}]\,(k^2\hat{t} - \tfrac{1}{2}k)\,\mathrm{e}^{k\hat{t}}\left(\frac{9\alpha}{2k}\right)^{1/2} \mathrm{erf}\,(k\hat{t})^{1/2} \\
& + \left(\frac{9\alpha}{2\pi}\right)^{1/2} k[(1-\alpha)\,\dot{\boldsymbol{u}}(0) - \boldsymbol{G}]\,\hat{t}^{1/2} \\
& - (1-\alpha)\,k^2\,\mathrm{e}^{k\hat{t}}\left[\int_0^{\hat{t}} \tau\,\mathrm{e}^{-k\tau}\dot{\boldsymbol{u}}(\tau)\,\mathrm{d}\tau - \hat{t}\int_0^{\hat{t}} \mathrm{e}^{-k\tau}\dot{\boldsymbol{u}}(\tau)\,\mathrm{d}\tau\right] \\
& + (1-\alpha)\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2\,\mathrm{e}^{k\hat{t}}\hat{t}\int_0^{\hat{t}}\left(\mathrm{e}^{-k\tau}\int_0^{\tau}\frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}}\right)\mathrm{d}\tau \\
& + (1-\alpha)\,k\,\mathrm{e}^{k\hat{t}}\left[\int_0^{\hat{t}} \tau\,\mathrm{e}^{-k\tau}\ddot{\boldsymbol{u}}(\tau)\,\mathrm{d}\tau - \hat{t}\int_0^{\hat{t}} \mathrm{e}^{-k\tau}\ddot{\boldsymbol{u}}(\tau)\,\mathrm{d}\tau\right] \\
& - (1-\alpha)\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2\,\mathrm{e}^{k\hat{t}}\int_0^{\hat{t}}\left(\tau\,\mathrm{e}^{-k\tau}\int_0^{\tau}\frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}}\right)\mathrm{d}\tau.
\end{aligned} \tag{3.17}$$

Equation (3.17) is the general solution for the Stokes motion of a particle starting at zero relative velocity, and with critical fluid-to-particle density ratio ($\alpha = 8/5$). Because (3.17) is only valid for the critical density ratio, its practical interest is very limited. It is shown here only for mathematical completeness, and will not be used to find the solution for specific background flows in §4. The transition from less than critical to larger than critical density ratio is smooth and the solution for the critical density ratio can be approached from either side of the spectrum of α. The critical value of $\alpha = 8/5$ is thus only of mathematical relevance and does not imply a change of physical character of the problem.

3.3. *The case* $\Delta < 0$ ($\alpha < 8/5$)

When the discriminant of the characteristic equation associated with (3.6) is smaller than zero, the roots of the characteristic equation are complex. The form of the real-valued solution to the homogeneous equation is

$$\boldsymbol{w}_h(\hat{t}) = \boldsymbol{A}\,\mathrm{e}^{-b\hat{t}/2}\cos(|\Delta|^{1/2}\,\hat{t}/2) + \boldsymbol{B}\,\mathrm{e}^{-b\hat{t}/2}\sin(|\Delta|^{1/2}\,\hat{t}/2), \tag{3.18}$$

with an associate Wronskian given by

$$W_{1,2}(\hat{t}) = \tfrac{1}{2}|\Delta|^{1/2}\,\mathrm{e}^{-b\hat{t}}. \tag{3.19}$$

Following the same procedure used in the previous subsections, the general solution for a particle heavier than critical is given by

$$\begin{aligned}
\boldsymbol{w}(\hat{t}) = {} & \boldsymbol{G} + \frac{2\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}}[\boldsymbol{G} - P\dot{\boldsymbol{u}}(0)]\,k\sin\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right) \\
& - \mathrm{e}^{-b\hat{t}/2}\left[\cos\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right) + \frac{b}{|\Delta|^{1/2}}\sin\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\right]\boldsymbol{G} \\
& + \frac{2\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}}\sin\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\left\{\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2[(1-\alpha)\,\dot{\boldsymbol{u}}(0) - \boldsymbol{G}]\int_0^{\hat{t}} \frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}}\cos\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau\right. \\
& \left. + \int_0^{\hat{t}} \mathrm{e}^{b\tau/2}\cos\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)(1-\alpha)\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2\int_0^{\tau}\frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}} - k\ddot{\boldsymbol{u}} - k^2\dot{\boldsymbol{u}}\right)\mathrm{d}\tau\right\} \\
& - \frac{2\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}}\cos\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\left\{\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2[(1-\alpha)\,\dot{\boldsymbol{u}}(0) - \boldsymbol{G}]\int_0^{\hat{t}} \frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}}\sin\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau\right. \\
& \left. + \int_0^{\hat{t}} \mathrm{e}^{b\tau/2}\sin\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)(1-\alpha)\left(\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2\int_0^{\tau}\frac{\ddot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma}{(\tau-\sigma)^{1/2}} - k\ddot{\boldsymbol{u}} - k^2\dot{\boldsymbol{u}}\right)\mathrm{d}\tau\right\}.
\end{aligned} \tag{3.20}$$

4. Application of (3.14) and (3.20) to specific flow fields

In this section, we apply the general solutions (3.14) and (3.20) to specific background flows. These background flow fields were selected to illustrate the applicability of the solutions, and to show some of the properties of the solutions. The three background velocity fields studied here are $\boldsymbol{u}(\hat{t}) = 0$, $\boldsymbol{u}(\hat{t}) = \boldsymbol{p}\hat{t}$, and $\dot{\boldsymbol{u}}(\hat{t}) = \boldsymbol{q}\delta(\hat{t} - 0^{+})$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are constant vectors. The first velocity field corresponds to a quiescent fluid, in which the particle is allowed to move under the influence of a constant gravitational field. The motion of a particle under these conditions has been determined before (Clift *et al.* 1978), and the solution derived by the method presented in this work is compared to the previous studies. The second velocity field corresponds to a fluid accelerating linearly in time (constant acceleration $\boldsymbol{p}$). This flow field is of particular interest here because it allows the formal solution given by (3.14) and (3.20) to be compared with the solution for the quiescent case after a Galileian transformation. The velocity of a particle in this case is equivalent to that related to the gravitationally induced motion if an appropriate reference frame is chosen. The third velocity field corresponds to an impulsive start, where the fluid is at rest at $\hat{t} = 0$, but jumps to a constant velocity $\boldsymbol{q}$ at $\hat{t} = 0^{+}$. The solution for this case is found after consideration of the terms related to the first and second derivatives of the fluid velocity in (3.14) and (3.20), and can be compared to asymptotic solutions found in the

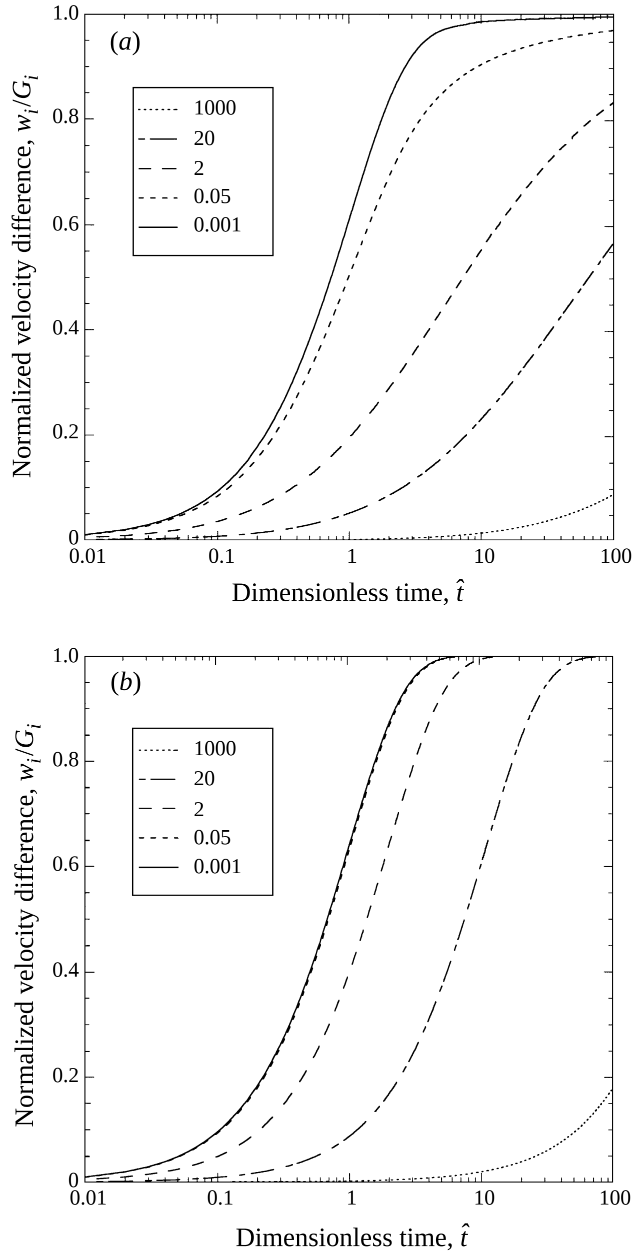

FIGURE 2. (*a*) Component of the normalized velocity difference in a generic direction $\boldsymbol{i}$ for different values of α and for the gravitationally induced motion (including history effects). (*b*) As (*a*) but neglecting history effects.

literature (Felderhof 1991). Felderhof made use of short- and long-term expansions to study the behaviour of a small particle under the action of an impulsive force. The solutions for these three simple cases require the use of all terms in (3.14) and (3.20), and may be superposed to describe more complex flow fields.

4.1. *Gravitationally induced flow in a quiescent fluid*

In this subsection, we study the solutions (3.14) and (3.20) for the case $\boldsymbol{u}(\hat{t}) = 0$. For this case, and for a particle lighter than critical ($\alpha > 8/5$), equation (3.14) is greatly simplified to yield

$$\boldsymbol{w}(\hat{t}) = \boldsymbol{v}(\hat{t}) = \left[1 + \left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erfc}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erfc}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\right]\boldsymbol{G}, \tag{4.1}$$

which is the solution previously found by Boggio (1927) and Sy *et al.* (1970) through the Laplace transformation of the equation of motion for a particle in a quiescent fluid. Because of the linearity of the equation of motion in uniform flows and at zero particle Reynolds number (equation (2.3)), this solution can always be added to solutions found for zero-gravity flows in order to find the velocity of a particle under the influence of a constant gravitational field.

For a particle with density larger than critical ($\alpha < 8/5$), equation (3.20) is reduced to

$$\begin{aligned}\boldsymbol{w}(\hat{t}) = \boldsymbol{v}(\hat{t}) = &\left[1 + \frac{(2k-b)\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}} \sin\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right) - \mathrm{e}^{-b\hat{t}/2} \cos\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\right]\boldsymbol{G} \\ &- \frac{2\boldsymbol{G}\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}} \sin\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2 \int_0^{\hat{t}} \frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}} \cos\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau \\ &+ \frac{2\boldsymbol{G}\,\mathrm{e}^{-b\hat{t}/2}}{|\Delta|^{1/2}} \cos\left(\frac{|\Delta|^{1/2}\,\hat{t}}{2}\right)\left(\frac{9\alpha}{2\pi}\right)^{1/2} k^2 \int_0^{\hat{t}} \frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}} \sin\left(\frac{|\Delta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau. \end{aligned} \tag{4.2}$$

Equation (4.2) is equivalent to the solution obtained by Boggio (1927) and presented in Clift *et al.* (1978) in terms of the imaginary error function. The reader should note that Table 11.1 in Clift *et al.* (1978) contains several typographical errors.

At this point, it is interesting to investigate the effect of the history term on the velocity of the particle. When the history term is neglected, the solution to (2.11) is readily found, yielding the following relative velocity (the subscript *nh* stands for 'no history'):

$$\boldsymbol{w}_{nh}(\hat{t}) = (1 - \mathrm{e}^{-k\hat{t}})\,\boldsymbol{G} - Pk\,\mathrm{e}^{-k\hat{t}} \int_0^{\hat{t}} \mathrm{e}^{k\sigma}\dot{\boldsymbol{u}}(\sigma)\,\mathrm{d}\sigma. \tag{4.3}$$

For the gravitationally induced motion in a quiescent fluid, and neglecting the history term, the second term on the right-hand side of (4.3) is identically zero, and the particle velocity is given by a simple exponential decay. Figure 2(*a*) shows (4.1) and (4.2) for different values of α. Figure 2(*b*) shows (4.3), where the history drag is neglected, for the same values of α. In figure 2(*a*), the component of the particle velocity (or the relative velocity) in a generic ***i***-direction is normalized by the ***i***th component of the terminal velocity $\boldsymbol{G}$ of the particle, which depends on α (equation (2.9)). Figures 2(*a*) and 2(*b*) show that the complete solutions (4.1) and (4.2) deviate significantly from the simplified solution (4.3) when the value of α deviates from zero. For very light particles, (4.1) and (4.3) show that it takes several thousand characteristic times to approach the terminal velocity, but figure 3 shows that the way the terminal velocity is approached depends on whether or not the history term is considered. For heavy particles (with larger characteristic times), the influence of the history term is not very strong.

As a numerical example, a 100 μm diameter bubble of air takes approximately $2 \times 10^5\,\tau_p$ to reach 90 % of its terminal velocity when released from rest in glycerin.

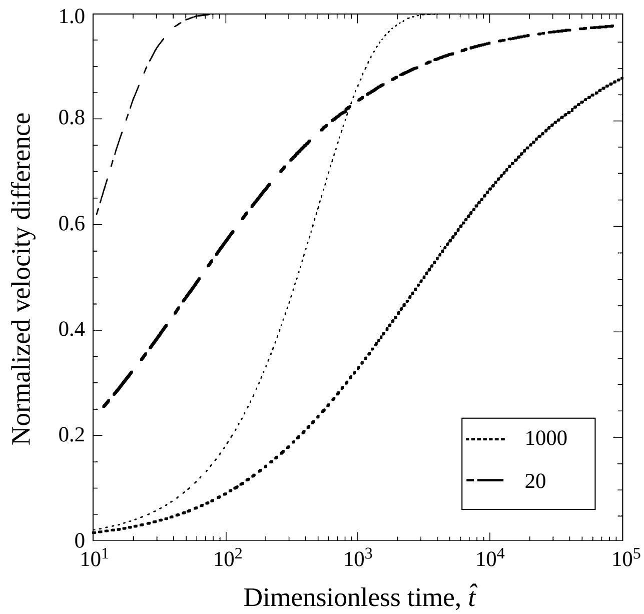

FIGURE 3. Component of the normalized velocity difference in a generic direction $\boldsymbol{i}$ for two different values of α and for the gravitationally induced motion (including history effects shown in thick lines and neglecting history effects in thin lines).

Even though τ_p of such air bubble is of the order of 10^{-9} s, the time to reach 90 % of its terminal velocity is calculated by (4.1) to be approximately 0.4 ms. Note that the added mass force strongly affects the response time of a bubble and for this reason the response time of bubbles is more closely represented by τ_p/k. Solution (4.3), which neglects the history drag contribution, gives a greatly underpredicted value of 2.2 μs. On the other hand, a particle 20 times denser than glycerin reaches 90 % of its terminal velocity in 9 times its characteristic time. The characteristic time τ_p of such 100 μm particle is of the order of 500 ms. Thus, this heavy particle also reaches 90 % of its terminal velocity in 0.4 ms. Solution (4.3) gives, for this case, 0.1 ms. In both situations the history drag contribution cannot be justifiably neglected, but the error is much larger in the case of the light particle.

It is important to note that the solution given by the momentum equation studied in this work predicts an approach to steady state as $t^{-1/2}$. The kernel of the history drag term and the respective approach to steady state in this equation is valid only for infinitesimal Reynolds number, since spherically symmetric diffusion of vorticity from the surface of the sphere is assumed in the derivation of equation (2.1). Higher-order corrections predict a faster approach to steady state and correspondingly different kernels for the history drag. As pointed out by Lovalenti & Brady (1993), the Oseen corrections become important for the long-term velocity behaviour of the particle for flows characterized by finite Re_p. However, in the first-order approximation, the contribution of the inertial terms is only felt after vorticity is diffused beyond the Oseen distance, which occurs for $t \sim \nu/|\boldsymbol{V}-\boldsymbol{U}|^2$. Due to the very low values of Re_p in the numerical examples given in the previous paragraph ($Re_p < 10^{-4}$), both bubble and heavy particle reach 90 % of their respective terminal velocities much before vorticity is diffused out from the particle surface to the Oseen distance. In this situation, the values calculated by the spherically symmetric kernel are correct.

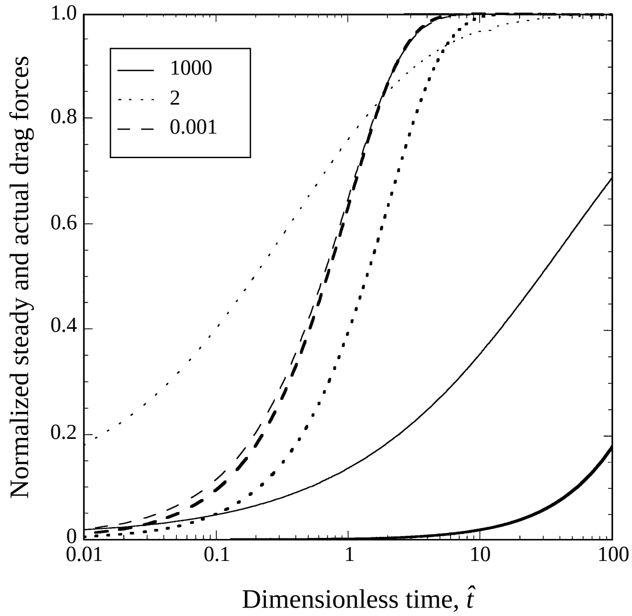

FIGURE 4. Normalized actual drag $w_i/G_i + (9\alpha/2\pi G_i^2)^{1/2}\int_0^{\hat{t}} \dot{w}_i\,\mathrm{d}\sigma/(\hat{t}-\sigma)^{1/2}$ (thick lines) and steady-state Stokes drag $w_{i,nh}/G_i$ when the history term is neglected (thin lines) for the gravitationally induced motion and for different values of α.

Figure 4 shows the comparison between the steady-state Stokes drag calculated by (4.3) and the actual drag that is the sum of the steady-state drag and the history drag as calculated by (4.1) and (4.2). It is clear from figure 4 that neglecting the history drag is not a reasonable assumption for light particles. Even for a particle that is only half as dense as the fluid, the actual drag is largely overpredicted by the steady-state Stokes drag for many hundreds of times the relaxation time of the particle. For a much heavier particle the history term contribution is of lesser importance. The steady-state Stokes drag in this case slightly overpredicts the actual drag for a time smaller than roughly the relaxation time of the particle and then underpredicts the actual drag for times larger than the relaxation time of the particle. It is important to note that, since the relative velocity potential in all curves is equal to the terminal velocity of the particle $\boldsymbol{G}$, the curves for the steady-state Stokes and the actual drag forces for a given value of α have to cross each other at a given point in time. The dimensionless time for which the steady-state drag shifts from overpredicting to underpredicting the magnitude of the actual drag is close to unity for very heavy particles and many orders of magnitude larger for light particles.

4.2. *Particle motion in a fluid undergoing constant acceleration*

The background velocity field in this case represents a good test of the general solutions (3.14) and (3.20) because it can be derived from (4.1) and (4.2) after physical reasoning. For the flow field $\boldsymbol{u}(\hat{t}) = \boldsymbol{p}\hat{t}$, we have $\dot{\boldsymbol{u}}(\hat{t}) = \boldsymbol{p}$ and $\ddot{\boldsymbol{u}}(\hat{t}) = 0$.

For a particle lighter than critical ($\alpha > 8/5$), evaluation of the two integrals related to $\dot{\boldsymbol{u}}(\hat{t})$ in (3.14) after rearrangement of the terms yields

$$\boldsymbol{w}(\hat{t}) = \left[1 + \left(\frac{9\alpha}{2\Delta}\right)^{1/2} k^2 \left(\frac{\mathrm{e}^{R\hat{t}}\,\mathrm{erfc}\,(R\hat{t})^{1/2}}{R^{1/2}} - \frac{\mathrm{e}^{Q\hat{t}}\,\mathrm{erfc}\,(Q\hat{t})^{1/2}}{Q^{1/2}}\right)\right][\boldsymbol{G} + (\alpha - 1)\boldsymbol{p}]. \quad (4.4)$$

Equation (4.4) indicates that, in a zero-gravity environment, the particle moves faster than the fluid with an asymptotic value of $(\alpha-1)\,\boldsymbol{p}$ for large times. This value is equivalent to the dimensionless terminal velocity $\boldsymbol{G}$ in the quiescent case when the reference frame is moving with the fluid. This result could be deduced without knowledge of (3.14), simply by considering a non-inertial reference frame and the solution for the quiescent case.

For a particle heavier than critical ($\alpha < 8/5$), equation (3.20) is reduced to

$$\begin{aligned}\boldsymbol{w}(\hat{t}) &= \left[1+\frac{(2k-b)\,\mathrm{e}^{-b\hat{t}/2}}{|\varDelta|^{1/2}}\sin\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)-\mathrm{e}^{-bt/2}\cos\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)\right][\boldsymbol{G}+(\alpha-1)\,\boldsymbol{p}]\\ &\quad -\frac{2[\boldsymbol{G}+(\alpha-1)\,\boldsymbol{p}]\,\mathrm{e}^{-b\hat{t}/2}}{|\varDelta|^{1/2}}\sin\left(\frac{|\varDelta|^{1/2}}{2}\right)\left(\frac{9\alpha}{2\pi}\right)^{1/2}k^2\int_0^{\hat{t}}\frac{\mathrm{e}^{b\sigma/2}}{\sigma^{1/2}}\cos\left(\frac{|\varDelta|^{1/2}\,\sigma}{2}\right)\mathrm{d}\sigma\\ &\quad +\frac{2[\boldsymbol{G}+(\alpha-1)\,\boldsymbol{p}]\,\mathrm{e}^{-b\hat{t}/2}}{|\varDelta|^{1/2}}\cos\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)\left(\frac{9\alpha}{2\pi}\right)^{1/2}k^2\int_0^{\hat{t}}\frac{\mathrm{e}^{b\sigma/2}}{\sigma^{1/2}}\sin\left(\frac{|\varDelta|^{1/2}\,\sigma}{2}\right)\mathrm{d}\sigma,\end{aligned} \tag{4.5}$$

where the same observations made in reference to (4.4) apply, with the difference that now the particle lags behind the fluid with an asymptotic value of $(\alpha-1)\boldsymbol{p}$ for values of α smaller than 1, and accelerates faster than the fluid with the same asymptotic value for values of α between 1 and 8/5. As expected, (4.4) and (4.5) are equivalent to (4.1) and (4.2). Figures 2 and 3 can be used for a combination of both situations by replacing the terminal velocity $\boldsymbol{G}$ by the vector $\boldsymbol{G}+(\alpha-1)\,\boldsymbol{p}$.

4.3. *Impulsive start at* $\hat{t} = 0^+$

In this case, the fluid undergoes an impulsive start at $\hat{t} = 0^+$, in the absence of a gravity field. This problem has been studied in connection with the Langevin equation and the Brownian motion of dilute suspensions (Hinch 1975), and was also approached more recently through the use of an asymptotic analysis by Felderhof (1991). In the case under study here, we have

$$\boldsymbol{u}(0)=\dot{\boldsymbol{u}}(0)=\ddot{\boldsymbol{u}}(0)=0, \quad \text{and} \quad \dot{\boldsymbol{u}}(0^+)=\boldsymbol{q}\delta(\hat{t}-0^+), \ddot{\boldsymbol{u}}(0^+)=\boldsymbol{q}\dot{\delta}(\hat{t}-0^+).$$

For a particle lighter than critical ($\alpha < 8/5$), all terms multiplying $\boldsymbol{G}$ and $\dot{\boldsymbol{u}}(0)$ are zero in (3.14), and the terms in square brackets of (3.14) yield the following relative velocity for $\hat{t} \geqslant 0^+$:

$$\boldsymbol{w}(\hat{t}) = \frac{(1-\alpha)\,k\boldsymbol{q}}{Q^{1/2}-R^{1/2}}[\mathrm{e}^{R\hat{t}}R^{1/2}\,\mathrm{erfc}\,(R\hat{t})^{1/2}-\mathrm{e}^{Q\hat{t}}Q^{1/2}\,\mathrm{erfc}\,(Q\hat{t})^{1/2}]. \tag{4.6}$$

For a particle heavier than critical ($\alpha > 8/5$), equation (3.20) gives

$$\begin{aligned}\boldsymbol{w}(\hat{t}) &= \left(\frac{9\alpha}{8\pi}\right)^{1/2}k^2\,\boldsymbol{q}(1-\alpha)\,\mathrm{e}^{-b\hat{t}/2}\\ &\quad\times\left\{\left[|\varDelta|^{1/2}\sin\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)+b\cos\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)\right]\int_0^{\hat{t}}\frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}}\sin\left(\frac{|\varDelta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau\right.\\ &\quad\left.-\left[b\sin\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)-|\varDelta|^{1/2}\cos\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)\right]\int_0^{\hat{t}}\frac{\mathrm{e}^{b\tau/2}}{\tau^{1/2}}\cos\left(\frac{|\varDelta|^{1/2}\,\tau}{2}\right)\mathrm{d}\tau\right\}\\ &\quad+\left[\frac{(2k-b)\,\mathrm{e}^{-b\hat{t}/2}}{|\varDelta|^{1/2}}\sin\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)-\mathrm{e}^{-b\hat{t}/2}\cos\left(\frac{|\varDelta|^{1/2}\,\hat{t}}{2}\right)\right]\boldsymbol{q}k(1-\alpha).\end{aligned} \tag{4.7}$$

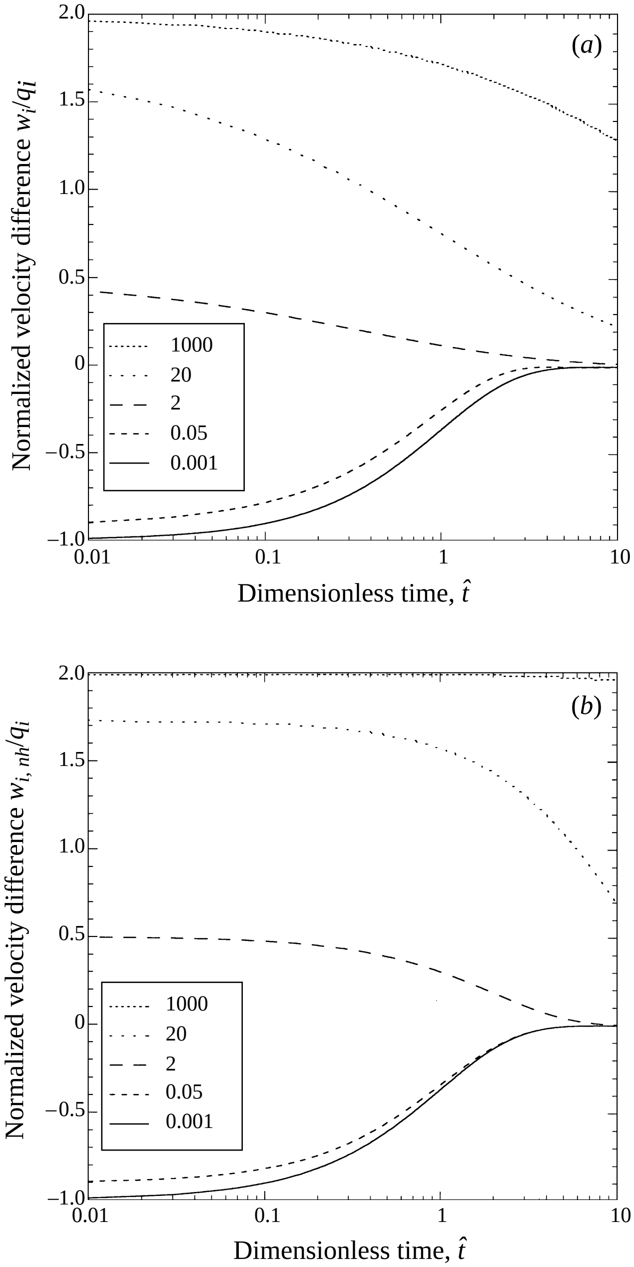

FIGURE 5. (*a*) Component of the normalized velocity difference in a generic direction $\boldsymbol{i}$ for different values of α and for the impulsive start motion (including history effects). (*b*) As (*a*) but neglecting history effects.

Figure 5(*a*) shows the behaviour of the normalized component of the relative velocity ($\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{q}$) in a generic $\boldsymbol{i}$-direction as given by (4.6) and (4.7) for different values of α. Figure 5(*b*) shows the same plot for the case where the history term is neglected. In this case, (4.3) yields

$$\boldsymbol{w}_{nh}(\hat{t}) = \boldsymbol{q}(\alpha - 1)\, k \exp(-k\hat{t}). \tag{4.8}$$

In all three solutions, the relative velocity at $t = 0^+$ is $\boldsymbol{q}(\alpha - 1)\, k$. The initial jump in the relative velocity indicates that a heavy particle has zero velocity at $t = 0^+$, but an infinitely light particle assumes a velocity three times the velocity of the fluid at $t = 0^+$. The impulsive start motion is also interesting because the response time is much shorter. The particles asymptote to the fluid velocity $\boldsymbol{q}$ much faster than the time required to reach the terminal velocity in the gravitationally induced motion. In fact, for values of α of order one or smaller, the particles approach the fluid velocity $\boldsymbol{q}$ in a time of the order of ten characteristic times.

In figure 5 we opted for not normalizing the relative velocity by the initial velocity jump as has been done in the literature (Felderhof 1991), although the results presented here validate the short- and long-term expansions presented by Felderhof (1991). The reason for not doing so is that this normalization leads to an artificial crossing of the velocity curves for different values of α when the history drag is considered. The crossing of the curves is due to the fact that a higher initial velocity does not characterize a more massive particle. This is because the initial jump in velocity is a function of α, and the mass of inertia depends on the density of the particle only. The crossing trajectory is thus another qualitative difference between the consideration or not of the history effects, but due only to a particular normalization choice. It is, however, important to emphasize that the time to reach any fraction of the velocity step $\boldsymbol{q}$ in the background flow is the same for a given particle, regardless of the magnitude of the step. This is analogous to saying that the time to reach any fraction of the terminal velocity $\boldsymbol{G}$ in the gravitationally induced motion is the same for a given particle, regardless of the magnitude of the gravity field.

As in the case of the gravitationally induced motion, the light particles are greatly affected by the inclusion of the history term. However, in contrast to what is observed in the gravitationally induced motion, figures 6(a) and 6(b) show that, for short times, the magnitude of the actual drag force (the sum of the fourth and fifth terms in equation (2.3)) is greatly underpredicted by the steady-state Stokes drag obtained by neglecting the history term ($\boldsymbol{w}_{nh}$). For the case of very light particles, the combined effect of a large history coefficient H and a non-zero relative velocity at $t = 0^+$ makes the value of the actual drag many orders of magnitude larger than the drag that is calculated by neglecting the history term contribution.

The fact that the actual drag is initially underpredicted in the impulsive start motion as opposed to being initially over predicted in the gravitationally induced motion is due to the different roles that the drag force plays in these flows. In the gravitationally induced motion, both steady-state Stokes and history drags are preventing the particle to reach terminal velocity. Neglecting the additional contribution of the history drag is equivalent to allowing the particle to approach its terminal velocity faster for initial times. In the case of the impulsive start, both contributions of the actual drag are the cause of the motion of the particle. Neglecting the history contribution is equivalent to make the particle initially go more slowly to the zero relative velocity. In both cases, the situation is reversed for long times since the relative velocity potential is the same, as discussed in §4.1.

The time required for an air bubble in water to reach 10% of its initial relative velocity $\boldsymbol{q}(\alpha-1)\,k$ is calculated by equation (4.6) to be approximately 645 characteristic times. Equation (4.8), which neglects the history term contribution, gives approximately 1,115 characteristic times, showing that the history term is responsible for shortening the response time of the bubble almost by half of the value given by the simplified solution (4.8). The difference in response times between the complete solutions (4.6) and (4.8) is not as large as it is in the gravitationally induced motion, but it is still very significant. The ratio of the simplified to the exact response times

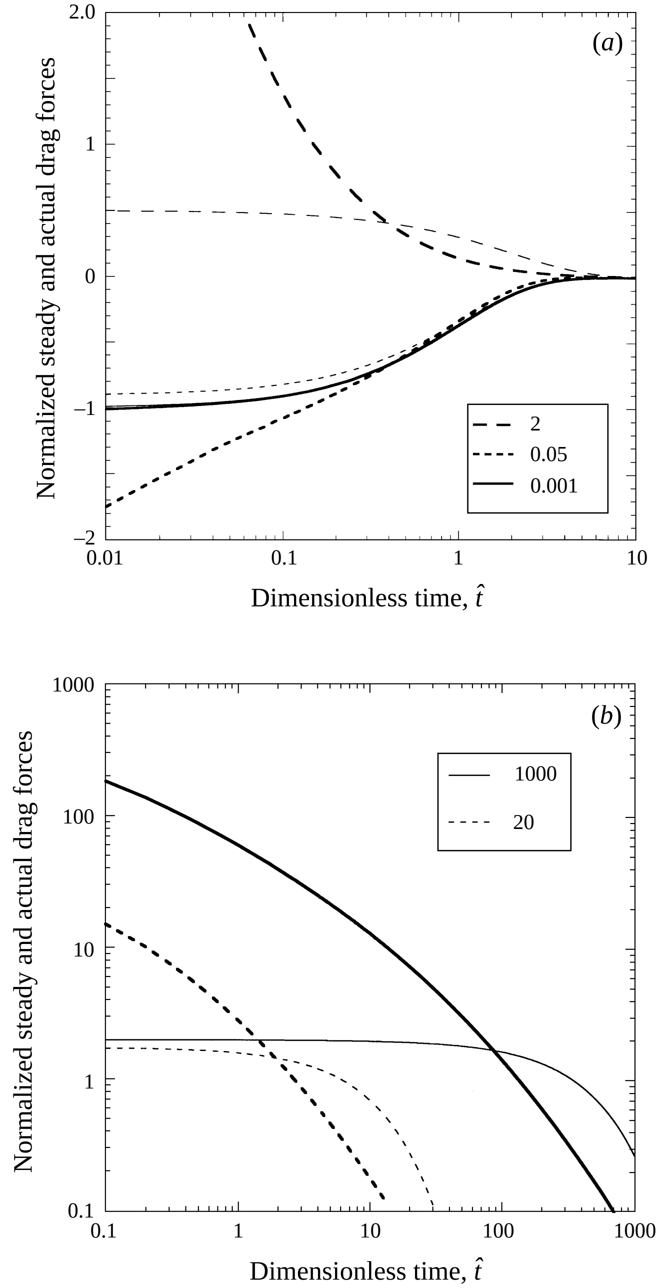

FIGURE 6. (*a*) Normalized actual drag $w_i/q_i+(9\alpha/2\pi q_i^2)^{1/2}\int_0^{\hat{t}}\dot{w}_i\,\mathrm{d}\sigma/(\hat{t}-\sigma)^{1/2}$ (thick lines) and steady-state Stokes drag $w_{i,nh}/q_i$ when the history term is neglected (thin lines) for the impulsive start motion and for different values of α. (*b*) As (*a*) but for higher values of α.

decreases for smaller fluid-to-particle density ratio. A particle 20 times denser than the fluid reaches 10 % of its initial relative velocity in 1.6 characteristic times when the history term is considered (solution (4.7)). Solution (4.8) gives 2.3 characteristic times for the same situation. The ratio of response times is, however, related to the fraction of the initial relative velocity that is chosen to define the response time. In the case of the impulsive motion, 10 % of the initial relative velocity of an air bubble in water is reached in a value of $\hat{t}$ that is much larger than the time for which the Stokes drag becomes larger than the actual drag (figure 6*b*). The value of $\hat{t}$ for which the drag curves cross in this case is 87.5, and 10% of the initial relative velocity is reached for a value of $\hat{t}$ that is an order of magnitude larger. In the gravitationally induced motion, 90% of the terminal velocity of an air bubble is reached much earlier than the time required for the crossing of the drag curves.

Figure 6(*a*) also shows that the history term contribution is negligible for a heavy particle. The fact that the history term in the equation of motion has negligible influence on the motion of heavy particles subjected to steady-state background flow fields is well established. Figure 6(*a*) shows that this is also the case for unsteady background flows when the relative acceleration asymptotes to zero for long times. This observation has important implications for the study of turbulent particle dispersion. In fact, the dimensionless time $\hat{t}$ in figure 6(*a*) is equivalent to the reciprocal of the Stokes number in an eddy–particle interaction in turbulent flows. The Stokes number is defined as the ratio of particle to fluid relaxation times. Characteristic Stokes numbers for heavy particles affected by turbulent flows are in the range of 0.1 to 10. Heavy particles interacting with turbulent flows with characteristic Stokes numbers smaller than 0.1 follow the instantaneous fluid velocity of the flow with negligible lagging (Wang & Stock 1992). For higher Stokes numbers, the heavy particles do not respond to turbulent fluctuations of the flow field, and the particles behave approximately as if the flow has no instantaneous fluctuations but only time-averaged velocity. These particles are mostly affected by unsteady large-eddy phenomena. Figure 6(*a*) shows that for an eddy–particle interaction, the history term does not play an important role for Stokes numbers varying from 0.1 to 100. Since the unsteady effects are more important for lower Stokes numbers, and figure 6(*a*) shows that for a purely unsteady flow (impulsive start) the history term is negligible for any value of Stokes number lower than 0.1, it is concluded that the history term contribution is unimportant for the modelling of turbulent particle dispersion of heavy particles. This conclusion supports the study by Mel, Adrian & Hanratty (1991). They studied the effect of the history drag on the turbulent diffusivity of a small heavy particle in a homogeneous, isotropic field. They concluded that the history term has negligible influence on the long-term particle dispersion mechanism or on the detailed structure of the time history of the fluid velocity seen by such particle.

5. Concluding remarks

The general solution of the particle equation of motion in unsteady Stokes flows has been obtained. The procedure to derive the solution consists of applying a linear operator to the equation of motion that serves the purpose of stretching the non-integer time derivative present in the history term into an integer derivative in a higher-order explicit equation. This higher-order equation is then solved exactly by the method of variation of parameters. The general solution is characterized by three different mathematical behaviours, corresponding to values of the fluid-to-particle density ratio α smaller than, equal to, and larger than the critical value of 8/5. The solution depends only on the particle and fluid parameters and on the derivatives of the background fluid velocity. The general solution found through the method outlined above was then applied to three different flow configurations, in order to illustrate its use and to validate the solution against previous results. The application of the solution to specific flow field configurations shows that neglecting the history term contribution may lead to inaccuracies when the particle is not much denser than the fluid.

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